Vector Space Definition

A vector space over a field F is a set V together with two operations, vector addition and scalar multiplication, satisfying the following properties:

1. Closure under addition: For all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} + \mathbf{w} \in V$.
2. Associativity of addition: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. Commutativity of addition: For all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
4. Existence of additive identity: There exists a vector $\mathbf{0} \in V$ such that for all $\mathbf{v} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
5. Existence of additive inverse: For every $\mathbf{v} \in V$, there exists a vector $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
6. Closure under scalar multiplication: For all $\mathbf{v} \in V$ and $\alpha \in F$, $\alpha\mathbf{v} \in V$.
7. Distributivity of scalar sums: For all $\mathbf{v} \in V$ and $\alpha, \beta \in F$, $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.
8. Distributivity of vector sums: For all $\mathbf{v}, \mathbf{w} \in V$ and $\alpha \in F$, $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$.
9. Compatibility of scalar multiplication with field multiplication: For all $\mathbf{v} \in V$ and $\alpha, \beta \in F$, $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$.
10. Identity element of scalar multiplication: For all $\mathbf{v} \in V$, $1 \cdot \mathbf{v} = \mathbf{v}$.

Example

Let $V = R^3$, the set of all real triples, and let $F = R$. Define vector addition and scalar multiplication component-wise:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$
$$\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \alpha v_3)$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\alpha \in R$.

Subspace Definition

Let V be a vector space over a field F. A subset W of V is called a subspace of V if it satisfies the following properties:

1. W contains the zero vector $\mathbf{0}$ of V.
2. W is closed under addition: For all $\mathbf{v}, \mathbf{w} \in W$, $\mathbf{v} + \mathbf{w} \in W$.
3. W is closed under scalar multiplication: For all $\alpha \in F$ and $\mathbf{v} \in W$, $\alpha\mathbf{v} \in W$.

Example

Consider the vector space $V = R^3$ over the field R. Let W be the subset of V defined by $W = \{(x, y, 0) \mid x, y \in R\}$.

To show that W is a subspace of V, we need to verify the three properties:

1. The zero vector $\mathbf{0} = (0, 0, 0)$ is in W.
2. Let $\mathbf{v} = (x_1, y_1, 0)$ and $\mathbf{w} = (x_2, y_2, 0)$ be arbitrary vectors in W. Then $\mathbf{v} + \mathbf{w} = (x_1 + x_2, y_1 + y_2, 0)$ is also in W.
3. Let $\alpha \in R$ and $\mathbf{v} = (x, y, 0)$ be an arbitrary vector in W. Then $\alpha\mathbf{v} = (\alpha x, \alpha y, 0)$ is also in W.

Since W satisfies all three properties, it is a subspace of V.

Example: Vector Space

Let $V = R^2$ be the vector space of all ordered pairs of real numbers with the usual operations of vector addition and scalar multiplication.

Example 1

Consider the vectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (-1, 5)$ in V. Determine if $\mathbf{u}$ and $\mathbf{v}$ span a subspace of V.

Solution

To check if the set $\{\mathbf{u}, \mathbf{v}\}$ spans a subspace of V, we need to verify if it satisfies the three properties of a subspace:

1. **Closure under addition:** For any $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v}$ must also be in V. Let's check:

$$\mathbf{u} + \mathbf{v} = (2, 3) + (-1, 5) = (2 - 1, 3 + 5) = (1, 8)$$

   Since $(1, 8)$ is in V, closure under addition holds.

2. **Closure under scalar multiplication:** For any scalar α and $\mathbf{u} \in V$, $\alpha\mathbf{u}$ must also be in V. Let's check with $\alpha = 2$:

$$2\mathbf{u} = 2(2, 3) = (4, 6)$$

   Since $(4, 6)$ is in V, closure under scalar multiplication holds.

3. **Contains the zero vector:** The zero vector in V is $(0, 0)$. Since $\mathbf{0} = (0, 0)$ is not in the set $\{\mathbf{u}, \mathbf{v}\}$, this set does not contain the zero vector.

Since the set $\{\mathbf{u}, \mathbf{v}\}$ fails to contain the zero vector, it does not form a subspace of V.

Basis and Dimension Definitions

Basis Definition

Let V be a vector space over a field F. A subset $\mathcal{B}$ of V is called a basis for V if the following conditions hold:

1. $\mathcal{B}$ spans V, i.e., every vector in V can be expressed as a linear combination of vectors in $\mathcal{B}$.
2. $\mathcal{B}$ is linearly independent, i.e., no vector in $\mathcal{B}$ can be expressed as a linear combination of the other vectors in $\mathcal{B}$.

Dimension Definition

The dimension of a vector space V, denoted as $\dim(V)$, is the number of vectors in any basis for V.

Example

Consider the vector space $V = R^3$ over the field R. Determine a basis for V and find the dimension of V.

Solution

To find a basis for V, we can start with the standard basis vectors:

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

We observe that any vector in R^3 can be written as a linear combination of these vectors. Thus, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ spans V.

Next, we need to check if these vectors are linearly independent. Suppose we have a linear combination of these vectors equal to the zero vector:

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = (0,0,0)$$

This equation leads to the system of equations:

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

which has only the trivial solution. Therefore, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent.

Since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is both spanning and linearly independent, it forms a basis for $V = R^3$. Thus, the dimension of V is 3.

Linear Transformations Definition

Let V and W be vector spaces over a field F. A function $T : V \to W$ is called a linear transformation if it satisfies the following two properties:

1. **Additivity:** For all $\mathbf{v}_1, \mathbf{v}_2 \in V$, $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$.
2. **Homogeneity:** For all $\mathbf{v} \in V$ and $\alpha \in F$, $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$.

Example

Consider the vector spaces $V = R^2$ and $W = R^2$, both over the field R. Define the transformation $T : V \to W$ by $T(\mathbf{v}) = A\mathbf{v}$, where A is the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Is T a Linear Transformation?

Additivity

Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be arbitrary vectors in V. Then,

$$\begin{aligned} T(\mathbf{v}_1 + \mathbf{v}_2) &= A(\mathbf{v}_1 + \mathbf{v}_2) \\ &= A\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + 2x_1 + x_2 \\ 3x_1 + 4x_2 + y_1 + 2y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &= T(\mathbf{v}_1) + T(\mathbf{v}_2) \end{aligned}$$

Hence, T satisfies the additivity property.

Homogeneity

Let $\alpha \in R$ and $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be an arbitrary vector in V. Then,

$$\begin{aligned} T(\alpha\mathbf{v}) &= A(\alpha\mathbf{v}) \\ &= A\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \\ &= \begin{pmatrix} \alpha x + 2\alpha y \\ 3\alpha x + 4\alpha y \end{pmatrix} \\ &= \alpha\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \alpha T(\mathbf{v}) \end{aligned}$$

Thus, T satisfies the homogeneity property.
Since T satisfies both additivity and homogeneity, it is a linear transformation.

Range and Null Space of a Linear Transformation

Let $T : V \to W$ be a linear transformation between vector spaces V and W over a field F.

Range Definition

The range of T, denoted as range(T) or Im(T), is the set of all possible outputs of T. It is defined as:

$$\text{range}(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$$

Null Space Definition

The null space of T, denoted as null(T) or ker(T), is the set of all vectors in V that map to the zero vector in W. It is defined as:

$$\text{null}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

Example

Consider the linear transformation $T : R^3 \to R^2$ defined by the matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$$

Range of T

To find the range of T, we need to determine all possible outputs of T. This is equivalent to finding all possible linear combinations of the columns of A. Since the columns of A are linearly dependent, the range of T will be a subspace of R^2 spanned by the column space of A.

$$\text{range}(T) = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right)$$

We can see that range(T) is the line in R^2 spanned by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Null Space of T

To find the null space of T, we need to find all vectors $\mathbf{v} \in R^3$ such that $A\mathbf{v} = \mathbf{0}$. This can be done by solving the homogeneous system $A\mathbf{v} = \mathbf{0}$.

$$A\mathbf{v} = \mathbf{0} \implies \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The augmented matrix of this system is row equivalent to:

$$\begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Solving this system yields:

$$x + 2y + z = 0$$

The solutions to this equation represent the null space of T. It is a plane in R^3.

Rank and Nullity Theorem

Let A be an $m \times n$ matrix over a field F.

Rank Definition

The rank of A, denoted as $\text{rank}(A)$, is the maximum number of linearly independent rows or columns of A. Equivalently, it is the dimension of the column space of A.

Nullity Definition

The nullity of A, denoted as $\text{nullity}(A)$, is the dimension of the null space of A, i.e., the number of free variables in the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Rank-Nullity Theorem

The rank-nullity theorem states that for any matrix A:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns of A.

Example

Consider the matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$$

Rank of A

To find the rank of A, we can perform row reduction:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that the second row is a multiple of the first row. So, $\text{rank}(A) = 1$.

Nullity of A

To find the nullity of A, we need to solve the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system has one free variable (e.g., we can choose $x = t$), so $\text{nullity}(A) = 1$.

Rank-Nullity Theorem

According to the rank-nullity theorem:

$$\text{rank}(A) + \text{nullity}(A) = 1 + 1 = 2$$

which is the number of columns of A.