Metric Space:

A metric space (X,d) consists of a set X and a metric $d : X \times X \to \mathbb{R}$ satisfying the following properties for all $x,y,z \in X$:

1. **Non-negativity:** $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$.
2. **Symmetry:** $d(x,y) = d(y,x)$.
3. **Triangle inequality:** $d(x,z) \leq d(x,y) + d(y,z)$.

Examples:

1. **Euclidean space:** Let $X = \mathbb{R}^n$ and $d(x,y) = \|x-y\|$, where $\|\cdot\|$ denotes the Euclidean norm.
2. **Discrete metric:** Let X be any set, and define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$.
3. **Taxicab metric:** Let $X = \mathbb{R}^2$ and $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

Open and Closed Balls in a Metric Space:

Let (X,d) be a metric space.

Open Ball:

For $x \in X$ and $r > 0$, the *open ball* centered at x with radius r is defined as

$$B(x,r) = \{y \in X \mid d(x,y) < r\}.$$

Closed Ball:

For $x \in X$ and $r > 0$, the *closed ball* centered at x with radius r is defined as

$$\overline{B}(x,r) = \{y \in X \mid d(x,y) \leq r\}.$$

Properties:

- The open ball $B(x,r)$ is an open set in X.
- The closed ball $\overline{B}(x,r)$ is a closed set in X.

Example:

Consider the Euclidean space $\mathbb{R}^n$ with the Euclidean metric. The open ball centered at a point $x \in \mathbb{R}^n$ with radius $r > 0$ is given by

$$B(x,r) = \{y \in \mathbb{R}^n \mid \|y-x\| < r\},$$

and the closed ball is given by

$$\overline{B}(x,r) = \{y \in \mathbb{R}^n \mid \|y-x\| \leq r\}.$$

Open Sets in a Metric Space:

Let (X,d) be a metric space.

A subset $U \subseteq X$ is called an *open set* if, for every point $x \in U$, there exists a positive real number $\varepsilon > 0$ such that the open ball $B(x,\varepsilon)$ is contained in U.

Examples:

1. **Open interval in $\mathbb{R}$:** Let $X = \mathbb{R}$ with the standard Euclidean metric. Then, for any $a,b \in \mathbb{R}$ with $a < b$, the open interval (a,b) is an open set.
2. **Open ball in Euclidean space:** Consider $\mathbb{R}^n$ with the Euclidean metric. For any $x \in \mathbb{R}^n$ and $r > 0$, the open ball $B(x,r)$ is an open set.
3. **Deleted open disk in $\mathbb{R}^2$:** Consider $\mathbb{R}^2$ with the Euclidean metric. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disk. The set $D \setminus \{(0,0)\}$ (deleted open disk) is an open set.

4. **Open set in discrete metric space:** Consider any set X with the discrete metric $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$.

   Then every subset of X is an open set.

Interior of a Set in a Metric Space:

Let (X,d) be a metric space.

The *interior* of a subset $A \subseteq X$, denoted by $\text{int}(A)$, is the largest open set contained in A. In other words, $\text{int}(A)$ consists of all points in A that are interior points of A.

Formally,

$$\text{int}(A) = \{x \in A \mid \text{there exists } r > 0 \text{ such that } B(x,r) \subseteq A\},$$

where $B(x,r)$ is the open ball centered at x with radius r.

Example:

Consider the set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. This set represents the open unit disk in $\mathbb{R}^2$. The interior of A, denoted by $\text{int}(A)$, is the set of points inside the disk, excluding the boundary. In this case,

$$\text{int}(A) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Limit Point of a Set in a Metric Space:

Let (X,d) be a metric space.

A point $x \in X$ is called a *limit point* of a subset $A \subseteq X$ if every open ball centered at x contains at least one point of A different from x itself.

Formally, x is a limit point of A if for every $\varepsilon > 0$, the open ball $B(x,\varepsilon)$ contains a point $y \in A$ such that $y \neq x$.

Example:

Consider the set $A = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$ in the metric space $(\mathbb{R}, |\cdot|)$. The point 0 is a limit point of A since for any $\varepsilon > 0$, the open ball $B(0,\varepsilon)$ contains infinitely many points of A. However, 0 is not an element of A.

Derived Set, Closed Set, Closure of a Set, Dense Set:

Let (X,d) be a metric space.

1. **Derived Set:** The derived set (also called the set of limit points or the set of accumulation points) of a subset $A \subseteq X$, denoted by A', is the set of all limit points of A.

2. **Closed Set:** A subset $A \subseteq X$ is called closed if it contains all its limit points. In other words, A is closed if $A' \subseteq A$.

3. **Closure of a Set:** The closure of a subset $A \subseteq X$, denoted by $\overline{A}$, is the smallest closed set containing A. It is the union of A and its derived set, i.e., $\overline{A} = A \cup A'$.

4. **Dense Set:** A subset $A \subseteq X$ is called dense in X if every point in X is either an element of A or a limit point of A. In other words, the closure of A is the entire space X, i.e., $\overline{A} = X$.

Examples:

1. Let $X = \mathbb{R}$ with the standard Euclidean metric.

   (a) The derived set of the set $A = (0,1) \cup \{2\}$ is $A' = [0,1]$.

   (b) The set $A = [0,1]$ is closed.

   (c) The closure of the set $A = (0,1)$ is $\overline{A} = [0,1]$.

   (d) The set $A = \mathbb{Q}$ (the set of rational numbers) is dense in $\mathbb{R}$.

2. Let $X = \mathbb{R}^2$ with the Euclidean metric.

   (a) The derived set of the set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (unit circle) is $A' = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

   (b) The set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ (closed unit disk) is closed.

(c) The closure of the set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ (open unit disk) is $\overline{A} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$.

(d) The set $A = \mathbb{Q}^2$ (the set of rational points in $\mathbb{R}^2$) is dense in $\mathbb{R}^2$.

Sequence and Subsequence in a Metric Space:
Let (X,d) be a metric space.

1. **Sequence:** A sequence in X is a function $f : \mathbb{N} \to X$, denoted by $\{x_n\}_{n=1}^{\infty}$, where $x_n = f(n)$ for each $n \in \mathbb{N}$. In other words, it is an infinite list of elements in X indexed by natural numbers.

2. **Subsequence:** A subsequence of a sequence $\{x_n\}_{n=1}^{\infty}$ is a sequence of the form $\{x_{n_k}\}_{k=1}^{\infty}$, where $n_1 < n_2 < n_3 < \dots$ are strictly increasing natural numbers.

Examples:

1. Consider the metric space $(\mathbb{R}, |\cdot|)$.

   (a) The sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = \frac{1}{n}$ for $n \geq 1$ converges to 0 as n approaches infinity.

   (b) A subsequence of $\{x_n\}_{n=1}^{\infty}$ can be obtained by selecting only those terms of the sequence for which n is a power of 2, i.e., $\{x_{2^k}\}_{k=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$.

2. Consider the metric space $(\mathbb{R}^2, d)$, where $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ is the Euclidean distance.

   (a) The sequence $\{(x_n,y_n)\}_{n=1}^{\infty}$ defined by $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$ for $n \geq 1$ converges to the point $(0,0)$ as n approaches infinity.

   (b) A subsequence of $\{(x_n,y_n)\}_{n=1}^{\infty}$ can be obtained by selecting only those terms of the sequence for which n is a prime number, i.e., $\{(x_{p_k},y_{p_k})\}_{k=1}^{\infty}$, where p_k is the k-th prime number.

Convergent Sequences, Cluster Points, and Cauchy Sequences in a Metric Space:
Let (X,d) be a metric space.

1. **Convergent Sequence:** A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to converge to a point $x \in X$ if for every $\varepsilon > 0$, there exists a positive integer N such that $d(x_n,x) < \varepsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.

2. **Cluster Point:** A point $x \in X$ is called a cluster point (or accumulation point) of a sequence $\{x_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to x.

3. **Cauchy Sequence:** A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called a Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer N such that $d(x_m,x_n) < \varepsilon$ for all $m,n \geq N$.

Examples:

1. **Convergent Sequence:**

   (a) In the metric space $(\mathbb{R}, |\cdot|)$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = \frac{1}{n}$ for $n \geq 1$ converges to 0 as n approaches infinity.

2. **Cluster Point:**

   (a) In the metric space $(\mathbb{R}, |\cdot|)$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = (-1)^n$ has two cluster points: -1 and 1.

3. **Cauchy Sequence:**

   (a) In the metric space $(\mathbb{R}, |\cdot|)$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = \frac{1}{n}$ for $n \geq 1$ is a Cauchy sequence.

Definition: A metric space (X,d) is called *complete* if every Cauchy sequence in X converges to a point in X.

Example: The real numbers $\mathbb{R}$ with the usual Euclidean distance $d(x,y) = |x-y|$ form a complete metric space.

Cantor's Intersection Theorem: Given a sequence of closed intervals $[a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \dots$ in $\mathbb{R}$, there exists a point x such that $x \in [a_n,b_n]$ for all $n \in \mathbb{N}$.

Example: Consider the sequence of closed intervals defined recursively as follows:

$$\begin{aligned}
[a_1,b_1] &= [0,1] \\
[a_2,b_2] &= \left[\frac{1}{2},1\right] \\
[a_3,b_3] &= \left[\frac{3}{4},1\right] \\
&\vdots
\end{aligned}$$

Each interval is a closed subinterval of the previous one, and their intersection is the singleton set $\{1\}$, which confirms Cantor's Intersection Theorem.

Theorem: (Cantor's Intersection Theorem) Given a sequence of closed intervals $[a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \dots$ in $\mathbb{R}$, there exists a point x such that $x \in [a_n,b_n]$ for all $n \in \mathbb{N}$.

Proof: Let $I_n = [a_n,b_n]$ be the sequence of nested closed intervals.

1. **Boundedness:** Since each I_n is a closed interval, they are bounded.

2. **Nested Intervals:** By construction, each interval contains the next one, i.e., $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$.

3. **Existence of a Point:** Let x be a point in $[a_1,b_1]$. Since $I_1 \supseteq I_2$, x is also in $[a_2,b_2]$. By induction, x is in every interval I_n, showing that x is in the intersection of all intervals.

Therefore, there exists a point x such that $x \in [a_n,b_n]$ for all $n \in \mathbb{N}$, thus confirming Cantor's Intersection Theorem.

Definition: Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Examples:

1. **Linear Function:** $f(x) = mx+b$ is continuous for any real numbers m and b.

2. **Polynomial Function:** $f(x) = x^2$ and $f(x) = x^3 + 2x^2 - 5$ are continuous over their entire domain, which is $\mathbb{R}$.

3. **Trigonometric Functions:** Functions like $f(x) = \sin(x)$, $f(x) = \cos(x)$, and $f(x) = \tan(x)$ are continuous over their respective domains.

4. **Exponential Function:** $f(x) = e^x$ (the natural exponential function) is continuous over the entire real line.

5. **Piecewise Continuous Function:** The function $f(x) = |x|$ is continuous everywhere except at $x = 0$.

Sequential Criterion: A function $f : X \to Y$ between topological spaces is continuous if and only if for every sequence (x_n) in X converging to x, the sequence $(f(x_n))$ converges to $f(x)$.

Other Characterizations of Continuity:

1. **Epsilon-Delta Definition:** $f : X \to Y$ is continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x,x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \varepsilon$.

2. **Preimage of Open Sets:** $f : X \to Y$ is continuous if the preimage of every open set in Y is an open set in X.

3. **Limit of Compositions:** Let $f : X \to Y$ and $g : Y \to Z$ be functions. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Examples:

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. It's continuous everywhere since it satisfies the epsilon-delta definition.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x)$. This function is continuous everywhere since it preserves sequential convergence.

3. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$. This function is discontinuous at $x = 0$ since the preimage of the open interval $(\frac{1}{2}, \frac{3}{2})$ is not open.

Uniform Continuity of Composite Functions: Let $f : X \to Y$ and $g : Y \to Z$ be functions between metric spaces. If f is uniformly continuous and g is continuous, then $g \circ f$ is uniformly continuous.

Proof Sketch: Let $\varepsilon > 0$ be given. Since g is continuous, for any $y_1, y_2 \in Y$, if $d_Y(y_1, y_2) < \delta_1$, then $d_Z(g(y_1), g(y_2)) < \varepsilon$. Similarly, since f is uniformly continuous, for the same ε, there exists $\delta_2 > 0$ such that if $d_X(x_1, x_2) < \delta_2$, then $d_Y(f(x_1), f(x_2)) < \delta_1$. Thus, for $d_X(x_1, x_2) < \delta_2$, we have $d_Z(g(f(x_1)), g(f(x_2))) < \varepsilon$, which shows that $g \circ f$ is uniformly continuous.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}$. f is continuous but not uniformly continuous, while g is uniformly continuous. However, their composition $g \circ f$ (which is $g(f(x)) = \sqrt{x^2} = |x|$) is uniformly continuous.

Homomorphism Definition: Let G and H be groups. A function $\varphi : G \to H$ is called a group homomorphism if for all $g_1, g_2 \in G$, $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$, where $\cdot$ denotes the group operation in G and H.

Example: Consider the function $\varphi : \mathbb{Z} \to \mathbb{Z}_6$ defined by $\varphi(n) = n \mod 6$. This function is a group homomorphism since for any $m, n \in \mathbb{Z}$, we have $\varphi(m+n) = (m+n) \mod 6 = (m \mod 6 + n \mod 6) \mod 6 = \varphi(m) + \varphi(n)$.

Example: Let $\varphi : \mathbb{R}^* \to \mathbb{R}^*$ be defined by $\varphi(x) = |x|$. This function is a group homomorphism since for any $x, y \in \mathbb{R}^*$, we have $\varphi(xy) = |xy| = |x| \cdot |y| = \varphi(x) \cdot \varphi(y)$.

Homomorphism Definition: Let G and H be groups. A function $\varphi : G \to H$ is called a group homomorphism if for all $g_1, g_2 \in G$, $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$, where $\cdot$ denotes the group operation in G and H.

Example: Consider the function $\varphi : (\mathbb{R}, +) \to (\mathbb{R}^*, \cdot)$ defined by $\varphi(x) = e^x$. This function is a group homomorphism since for any $x, y \in \mathbb{R}$, we have $\varphi(x+y) = e^{x+y} = e^x \cdot e^y = \varphi(x) \cdot \varphi(y)$.

Example: Let $\varphi : (\mathbb{Z}, +) \to (\mathbb{Z}_6, +)$ be defined by $\varphi(n) = n \mod 6$. This function is a group homomorphism since for any $m, n \in \mathbb{Z}$, we have $\varphi(m+n) = (m+n) \mod 6 = (m \mod 6 + n \mod 6) \mod 6 = \varphi(m) + \varphi(n)$.

Characterizations of Homomorphisms:

1. **Preservation of Identity:** A function $\varphi : G \to H$ between groups is a homomorphism if and only if $\varphi(e_G) = e_H$, where e_G and e_H are the identities in G and H respectively.

2. **Preservation of Inverses:** A function $\varphi : G \to H$ between groups is a homomorphism if and only if $\varphi(g^{-1}) = (\varphi(g))^{-1}$ for all $g \in G$.

Examples:

1. Consider the function $\varphi : (\mathbb{Z}, +) \to (\mathbb{Z}_6, +)$ defined by $\varphi(n) = n \mod 6$. This function is a group homomorphism since it preserves the identity: $\varphi(0) = 0 \mod 6 = 0$.

2. Let $\varphi : (\mathbb{R}^*, \cdot) \to (\mathbb{R}^*, \cdot)$ be defined by $\varphi(x) = x^2$. This function is a group homomorphism since it preserves inverses: $\varphi(x^{-1}) = (x^{-1})^2 = (x^2)^{-1} = (\varphi(x))^{-1}$.