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1 Introduction

Limit Definition:

The limit of a function $f(x)$ as x approaches a value c is denoted by:

$$\lim_{x \to c} f(x) = L$$

This means that as x gets arbitrarily close to c, the values of $f(x)$ get arbitrarily close to L.

Formally, we say that L is the limit of $f(x)$ as x approaches c if for every positive number ε, there exists a positive number δ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Example:

Consider the function $f(x) = \frac{1}{x}$ and the limit:

$$\lim_{x \to 2} \frac{1}{x}$$

To find this limit, we observe the behavior of $f(x)$ as x approaches 2. As x gets closer to 2, the values of $f(x)$ get larger. We can also observe this behavior from the right and left sides of 2:

$$\lim_{x \to 2^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 2^-} \frac{1}{x} = -\infty$$

Since the left-hand limit and the right-hand limit do not approach the same value, the limit $\lim_{x \to 2} \frac{1}{x}$ does not exist. **Epsilon-Delta Definition of a Limit:**

Let $f(x)$ be a function defined in an open interval containing c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is L, denoted by

$$\lim_{x \to c} f(x) = L$$

if for every positive number ε, there exists a positive number δ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In other words, for any given positive tolerance ε, we can find a positive number δ such that the distance between $f(x)$ and L is less than ε whenever x is within δ units of c, excluding c itself.

Example:

Consider the function $f(x) = 2x - 1$ and the limit

$$\lim_{x \to 2} (2x - 1)$$

To prove that the limit is 3, we need to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - 2| < \delta$, we have $|(2x - 1) - 3| < \varepsilon$.

Let's choose $\varepsilon = 0.1$. We want to find a δ such that $|(2x - 1) - 3| = |2x - 4| < 0.1$ whenever $0 < |x - 2| < \delta$.

If we choose $\delta = 0.05$, then whenever $0 < |x - 2| < 0.05$, we have $|2x - 4| = |2(x - 2)| = 2|x - 2| < 0.1$, which satisfies the condition.

Thus, by choosing $\varepsilon = 0.1$ and $\delta = 0.05$, we have shown that the limit $\lim_{x \to 2} (2x - 1) = 3$.

Example of Limit Problem:

Let's prove that the limit of the function $f(x) = 3x - 1$ as x approaches 2 is 5 using the epsilon-delta definition.

We want to show that for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - 2| < \delta$, we have $|(3x - 1) - 5| < \varepsilon$.

Given $|(3x - 1) - 5| < \varepsilon$, we simplify to $|3x - 6| < \varepsilon$, which further simplifies to $|3(x - 2)| < \varepsilon$.

We need to bound $|x - 2|$, so we choose $\delta = \frac{\varepsilon}{3}$.

Now, whenever $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$, we have $|3(x - 2)| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.

Thus, by choosing $\delta = \frac{\varepsilon}{3}$, we have shown that the limit $\lim_{x\to 2}(3x - 1) = 5$.

Continuity of Functions:

A function $f(x)$ is said to be continuous at a point c if the following conditions are met:

1. The function $f(x)$ is defined at c.
2. The limit $\lim_{x\to c} f(x)$ exists.
3. The value of the limit $\lim_{x\to c} f(x)$ equals $f(c)$.

If a function is continuous at every point in its domain, it is called a continuous function.

Types of Discontinuities:

- **Removable Discontinuity:** A removable discontinuity occurs when there is a hole or gap in the graph of the function that can be filled in by redefining the function at a single point.
- **Jump Discontinuity:** A jump discontinuity occurs when the left-hand and right-hand limits exist, but they are not equal.
- **Infinite Discontinuity:** An infinite discontinuity occurs when the function approaches positive or negative infinity as it approaches a point from both sides.

Example:

Consider the function $f(x) = \frac{x^2-1}{x-1}$.

This function is not defined at $x = 1$ since it results in division by zero. However, if we simplify the function, we get $f(x) = x + 1$, which is defined at $x = 1$ and equals 2.

Thus, the function $f(x)$ has a removable discontinuity at $x = 1$. **Example with Solution:**

Consider the function $f(x) = \sqrt{x}$.

To determine the continuity of $f(x)$, we need to check if it satisfies the conditions for continuity at all points in its domain.

Solution:

1. **Function Defined:** The function $f(x) = \sqrt{x}$ is defined for $x \geq 0$, which means it is defined at all points in its domain.

2. **Limit Exists:** Let's consider the limit $\lim_{x\to c} \sqrt{x}$ as x approaches any point c in its domain ($c \geq 0$).

If we approach c from the right side ($x > c$), we have:

$$\lim_{x\to c^+} \sqrt{x} = \sqrt{c}$$

If we approach c from the left side ($x < c$), we have:

$$\lim_{x\to c^-} \sqrt{x} = \sqrt{c}$$

So, the limit exists for all points c in the domain of $f(x)$.

3. **Value of Limit Equals $f(c)$:** For any point c in the domain of $f(x)$, we have $f(c) = \sqrt{c}$.

Therefore, $\lim_{x\to c} \sqrt{x} = f(c)$, satisfying the condition for continuity at all points in the domain.

Hence, the function $f(x) = \sqrt{x}$ is continuous for $x \geq 0$.

Differentiability:

A function $f(x)$ is said to be differentiable at a point c if the following limit exists:

$$f'(c) = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h}$$

If this limit exists, $f(x)$ is said to be differentiable at c, and $f'(c)$ is called the derivative of $f(x)$ at c.

Example 1:

Consider the function $f(x) = 3x^2 - 2x + 1$.
To find where $f(x)$ is differentiable, we need to compute its derivative $f'(x)$ using the limit definition.
Solution:
We have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 - 2(x+h) + 1 - (3x^2 - 2x + 1)}{h} \\
&= \lim_{h\to 0} \frac{3(x^2 + 2hx + h^2) - 2x - 2h + 1 - 3x^2 + 2x - 1}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6hx + 3h^2 - 2x - 2h + 1 - 3x^2 + 2x - 1}{h} \\
&= \lim_{h\to 0} \frac{6hx + 3h^2 - 2h}{h} \\
&= \lim_{h\to 0} (6x + 3h - 2) \\
&= 6x - 2
\end{aligned}$$

So, $f'(x) = 6x - 2$.
Since this derivative is a polynomial, it is defined for all real numbers x, and thus, $f(x)$ is differentiable everywhere.
Example 2:
Consider the function $f(x) = |x|$.
To determine where $f(x)$ is differentiable, let's compute its derivative using the limit definition.
Solution:
We have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{|x+h| - |x|}{h}
\end{aligned}$$

The behavior of $|x+h|$ depends on the sign of $x+h$. If $x+h$ is positive, then $|x+h| = x+h$. If $x+h$ is negative, then $|x+h| = -(x+h) = -x - h$.

Therefore, $|x+h| - |x| = (x+h) - x = h$ when $x+h$ is positive, and $|x+h| - |x| = -(x+h) - (-x) = -h$ when $x+h$ is negative.

So, the derivative $f'(x)$ will be different depending on the sign of x.

If $x > 0$, then:

$$f'(x) = \lim_{h\to 0^+} \frac{h}{h} = 1$$

If $x < 0$, then:

$$f'(x) = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

At $x = 0$, the function is not differentiable because the left-hand and right-hand limits of $f'(x)$ are different.

Therefore, $f(x)$ is differentiable everywhere except at $x = 0$

Uniform Continuity:

A function $f(x)$ is said to be uniformly continuous on an interval $[a, b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in $[a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

In other words, the choice of δ depends only on ε and not on the specific points x and y.

Example:

Consider the function $f(x) = \sqrt{x}$ defined on the interval $[0, 1]$.

We want to determine whether $f(x)$ is uniformly continuous on this interval.

Solution:

To prove uniform continuity, we need to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in $[0, 1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Since $f(x) = \sqrt{x}$ is a continuous function on the closed interval $[0, 1]$, it is uniformly continuous on this interval.

This is because for any $\varepsilon > 0$, we can choose $\delta = \varepsilon^2$ such that for all x, y in $[0, 1]$, if $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$. Thus, $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$.

Example 1:

Let $f(x) = 2x$ on the interval $[0, 1]$. We want to show that $f(x)$ is uniformly continuous on $[0, 1]$.

Solution:

Given any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$. Then, for any $x_1, x_2 \in [0, 1]$ such that $|x_1 - x_2| < \delta$, we have

$$|f(x_1) - f(x_2)| = |2x_1 - 2x_2| = 2|x_1 - x_2| < 2\delta = 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Therefore, $f(x)$ is uniformly continuous on $[0, 1]$.

Example 2:

Let $g(x) = \sin(x)$ on the interval $[0, \pi]$. We want to show that $g(x)$ is uniformly continuous on $[0, \pi]$.

Solution:

Given any $\epsilon > 0$, choose $\delta = \epsilon$. Then, for any $x_1, x_2 \in [0, \pi]$ such that $|x_1 - x_2| < \delta$, we have

$$|g(x_1) - g(x_2)| = |\sin(x_1) - \sin(x_2)| \leq |x_1 - x_2| < \delta = \epsilon.$$

Therefore, $g(x)$ is uniformly continuous on $[0, \pi]$.

Boundedness Theorem (Heine-Borel Theorem):

A subset E of R^n is compact if and only if it is closed and bounded.

Proof:

($\Rightarrow$) Suppose E is compact. Then E is closed and bounded.

($\Leftarrow$) Suppose E is closed and bounded. We want to show that E is compact.

Since E is bounded, there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in E$, where $\|\cdot\|$ denotes the Euclidean norm.

Now, let $\{x_k\}$ be any sequence in E. Since E is bounded, by Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{x_{k_j}\}$ of $\{x_k\}$. Let $x_{k_j} \to x$ as $j \to \infty$. Since E is closed, $x \in E$.

Therefore, E is sequentially compact, and by the equivalence of compactness and sequential compactness in R^n, E is compact.

Intermediate Value Theorem (IVT):

Let f be a continuous function on a closed interval $[a, b]$. If y_0 lies between $f(a)$ and $f(b)$, then there exists c in $[a, b]$ such that $f(c) = y_0$.

Proof:

Consider the function $g(x) = f(x) - y_0$. Since f is continuous on $[a, b]$, g is also continuous on $[a, b]$.

Observe that $g(a) = f(a) - y_0 < 0$ and $g(b) = f(b) - y_0 > 0$ since y_0 lies between $f(a)$ and $f(b)$.

By the Intermediate Value Theorem for continuous functions, there exists c in $[a, b]$ such that $g(c) = 0$, which implies $f(c) = y_0$.

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Extreme Value Theorem (EVT):

Let f be a continuous function on a closed interval $[a, b]$. Then, there exist points c and d in $[a, b]$ such that $f(c)$ is the maximum value of f on $[a, b]$ and $f(d)$ is the minimum value of f on $[a, b]$.

Proof:

Consider the set $S = \{f(x) : x \in [a, b]\}$, the range of f on $[a, b]$.

Since f is continuous on the closed interval $[a, b]$, by the Extreme Value Theorem for continuous functions, S is a closed and bounded set.

Therefore, S has a maximum element, say M, and a minimum element, say m.

By definition of S, there exist c and d in $[a, b]$ such that $f(c) = M$ and $f(d) = m$, respectively.

Thus, $f(c)$ is the maximum value of f on $[a, b]$, and $f(d)$ is the minimum value of f on $[a, b]$.

Darboux Intermediate Value Theorem (IVT):

Let f be a real-valued function defined on a closed interval $[a, b]$. If f is continuous on $[a, b]$, then for any y between $f(a)$ and $f(b)$, there exists c in (a, b) such that $f(c) = y$.

Proof:
Assume, without loss of generality, that $f(a) < f(b)$. Let y be a real number between $f(a)$ and $f(b)$.
Define $g(x) = f(x) - y$. Then $g(a) < 0$ and $g(b) > 0$.
Since f is continuous on $[a, b]$, g is also continuous on $[a, b]$.
By the Intermediate Value Theorem, there exists c in (a, b) such that $g(c) = 0$, which implies $f(c) = y$.
Chain Rule:
Let $f(x)$ and $g(x)$ be differentiable functions. If $y = f(g(x))$, then $y' = f'(g(x)) \cdot g'(x)$.
Proof:
Consider the function $h(x) = f(g(x))$.
By the definition of the derivative, we have

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x}.$$

Since f is differentiable at $g(x)$, we can write $f(g(x+\Delta x)) - f(g(x))$ as $f'(g(x)) \cdot g'(x) \cdot \Delta x + \epsilon(\Delta x)$, where $\epsilon(\Delta x)$ approaches 0 as Δx approaches 0.

Therefore,

$$h'(x) = f'(g(x)) \cdot g'(x) + \lim_{\Delta x \to 0} \frac{\epsilon(\Delta x)}{\Delta x}.$$

Since $\lim_{\Delta x \to 0} \frac{\epsilon(\Delta x)}{\Delta x}$ approaches 0 as Δx approaches 0, we have $h'(x) = f'(g(x)) \cdot g'(x)$, which proves the Chain Rule.

Definition:
A sequence is an ordered list of numbers written in a specific order. It can be finite or infinite.
Examples:

1. The sequence (a_n) defined by $a_n = n^2$ for $n = 1, 2, 3, \ldots$ is an example of an infinite sequence. Its terms are $1, 4, 9, 16, \ldots$.
2. The sequence (b_n) defined by $b_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$ is another example of an infinite sequence. Its terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.
3. The sequence (c_n) defined by $c_n = (-1)^n$ for $n = 1, 2, 3, \ldots$ is an example of an infinite sequence alternating between -1 and 1. Its terms are $-1, 1, -1, 1, \ldots$.
4. The sequence (d_n) defined by $d_n = \sin\left(\frac{n\pi}{2}\right)$ for $n = 1, 2, 3, \ldots$ is an example of an infinite sequence oscillating between -1 and 1 with period 4. Its terms are $1, 0, -1, 0, 1, 0, -1, 0, \ldots$.

Definition:
The limit of a sequence (a_n) as n approaches infinity, denoted by $\lim_{n\to\infty} a_n$, is the number L such that for every positive number ϵ, there exists a positive integer N such that $|a_n - L| < \epsilon$ for all $n > N$.
Examples:

1. Consider the sequence (a_n) defined by $a_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$. We claim that $\lim_{n\to\infty} a_n = 0$.

   Given $\epsilon > 0$, choose $N = \lceil \frac{1}{\epsilon} \rceil$. Then for all $n > N$, we have $|a_n - 0| = \frac{1}{n} < \frac{1}{N} \leq \epsilon$. Thus, $\lim_{n\to\infty} a_n = 0$.

2. Let (b_n) be the sequence defined by $b_n = (-1)^n$ for $n = 1, 2, 3, \ldots$. We claim that this sequence does not converge.

   Suppose for contradiction that $\lim_{n\to\infty} (-1)^n = L$. Then for every positive number ϵ, there exists N such that $|(-1)^n - L| < \epsilon$ for all $n > N$. However, notice that $(-1)^{N+1} - L > \epsilon$ and $(-1)^{N+2} - L < -\epsilon$, which is a contradiction since L cannot simultaneously be less than and greater than ϵ. Therefore, (b_n) does not converge.

3. Consider the sequence (c_n) defined by $c_n = \frac{n^2+3n+1}{n^2-2n+1}$ for $n = 1, 2, 3, \ldots$. We claim that $\lim_{n\to\infty} c_n = 1$.

   Rewrite c_n as $c_n = \frac{n^2(1+\frac{3}{n}+\frac{1}{n^2})}{n^2(1-\frac{2}{n}+\frac{1}{n^2})} = \frac{1+\frac{3}{n}+\frac{1}{n^2}}{1-\frac{2}{n}+\frac{1}{n^2}}$.

   As $n \to \infty$, $\frac{3}{n}$ and $\frac{2}{n}$ approach 0, and $\frac{1}{n^2}$ approaches 0 even faster. Therefore, $\lim_{n\to\infty} c_n = \frac{1+0+0}{1-0+0} = 1$.