IV SEM

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1 Introduction

Origin of a Second-Order PDE:

Consider a second-order partial differential equation (PDE) of the form:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g$$

where $u(x, y)$ is the unknown function, and a, b, c, d, e, f, g are given functions.

The origin of this PDE typically arises from physical or mathematical modeling of phenomena involving two independent variables, such as heat conduction, wave propagation, fluid dynamics, etc.

Examples of Partial Differential Equations (PDEs) with Solutions:

1. **Heat Equation:**
   - The heat equation in one dimension is given by:
     $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
     where $u(x, t)$ represents the temperature distribution, k is the thermal diffusivity, and t is time.
   - **Solution:** Depending on the boundary and initial conditions, solutions to the heat equation can be obtained using methods like separation of variables or Fourier series.

2. **Wave Equation:**
   - The wave equation in one dimension is given by:
     $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$
     where $u(x, t)$ represents the displacement of a wave, c is the wave speed, and t is time.
   - **Solution:** Solutions to the wave equation can be found using methods like separation of variables or Fourier transforms, depending on the boundary and initial conditions.

3. **Laplace's Equation:**
   - Laplace's equation in two dimensions is given by:
     $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
     where $u(x, y)$ represents a scalar field.
   - **Solution:** Solutions to Laplace's equation can be obtained using methods like separation of variables or the method of images, depending on the boundary conditions.

Types of Fractional Differential Equations (FDEs) with Examples:

1. **Fractional Ordinary Differential Equation (FODE):**

- Example: The fractional ordinary differential equation of order α is given by:

$$\frac{d^\alpha u}{dt^\alpha} = f(t, u, u', u'', \ldots, u^{(n)})$$

where $u(t)$ is the unknown function, f is a given function, and $u^{(k)}$ represents the kth derivative of u with respect to t.

2. **Fractional Partial Differential Equation (FPDE):**

- Example: The fractional partial differential equation of order α in one dimension is given by:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(t, x, u, u_t, u_{xx}, \ldots, u_{x^n})$$

where $u(x, t)$ is the unknown function, F is a given function, and u_{x^k} represents the kth derivative of u with respect to x.

3. **Caputo Fractional Differential Equation (CFDE):**

- Example: The Caputo fractional differential equation of order α is given by:

$$D^\alpha u(t) = f(t, u, u', u'', \ldots, u^{(n)})$$

where $u(t)$ is the unknown function, f is a given function, and D^α represents the Caputo fractional derivative.

Partial Differential Equation (PDE) of First Order and First Degree:

A partial differential equation (PDE) of first order and first degree in two variables x and y has the general form:

$$F(x, y, u, u_x, u_y) = 0$$

where $u(x, y)$ is the unknown function, and u_x and u_y represent the partial derivatives of u with respect to x and y respectively.

Example:

Consider the PDE:

$$u_x + u_y = xy$$

Solution:

To solve this PDE, we can use the method of characteristics. Let p and q be parameters along the characteristic curves defined by the system of ordinary differential equations:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = xy$$

Solving these equations, we find $x = t + p$, $y = t + q$, and $u = \frac{1}{2}t^2 + pt + pq$.

The general solution to the PDE is given implicitly by $u = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + C$, where C is an arbitrary constant.

Lagrange's Solution Method for First-Order Linear ODEs:

Consider a first-order linear ordinary differential equation (ODE) of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The solution can be found using Lagrange's method, which involves integrating the integrating factor $e^{\int P(x)\,dx}$ into both sides of the equation.

The solution is given by:

$$y(x) = \frac{1}{\mu(x)} \left(\int Q(x)\mu(x)\,dx + C \right)$$

where $\mu(x) = e^{\int P(x)\,dx}$ is the integrating factor and C is the constant of integration.

Example:
Consider the first-order linear ODE:

$$\frac{dy}{dx} + 2xy = x^2$$

Solution:
First, we find the integrating factor:

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

Now, multiply both sides of the equation by $\mu(x)$:

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = x^2e^{x^2}$$

This can be written as:

$$\frac{d}{dx}(e^{x^2}y) = x^2e^{x^2}$$

Integrating both sides with respect to x, we get:

$$e^{x^2}y = \int x^2e^{x^2}\,dx + C$$

Solving the integral, we have:

$$e^{x^2}y = \frac{1}{2}x^2e^{x^2} + C$$

Thus, the solution to the ODE is:

$$y(x) = \frac{1}{2}x^2 + Ce^{-x^2}$$

Charpit's Solution Method for First-Order PDEs:
Consider a first-order partial differential equation (PDE) of the form:

$$F(x, y, u, p, q) = 0$$

where $u(x, y)$ is the unknown function, and p and q are auxiliary variables.
Charpit's method involves solving the system of ordinary differential equations (ODEs):

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{F_u} = \frac{dp}{F_p} = \frac{dq}{F_q}$$

where a, b, F_u, F_p, F_q are partial derivatives of F with respect to their respective variables.
The solution $u(x, y)$ can then be expressed implicitly in terms of x, y, p, q using the relations obtained from integrating the ODEs.
Example:
Consider the first-order PDE:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Solution:
Using Charpit's method, we have the following system of ODEs:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u} = \frac{dp}{1} = \frac{dq}{1}$$

Solving these equations, we obtain the relations:

$$u = \frac{p}{q}, \quad p = C_1xy, \quad q = C_2$$

where C_1 and C_2 are constants of integration.

Thus, the solution to the PDE is given implicitly by:

$$u = \frac{C_1 xy}{C_2}$$

Origin of Second-Order Partial Differential Equations (PDEs):

Second-order PDEs commonly arise in various areas of mathematics and physics to describe phenomena involving rates of change or propagation of quantities that depend on two or more independent variables.

(a) **Wave Equation:** The wave equation describes the behavior of waves such as sound waves, electromagnetic waves, and seismic waves. It is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(b) **Heat Equation:** The heat equation describes the distribution of heat in a solid medium over time. It is given by:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

(c) **Laplace's Equation:** Laplace's equation is a special case of the Poisson equation and describes steady-state phenomena such as electrostatics and fluid flow. It is given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

These examples illustrate the diverse applications of second-order PDEs in modeling physical systems and processes.

Solution of Second-Order PDE with Constant Coefficients:

Consider a second-order partial differential equation (PDE) with constant coefficients:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = F(x, y)$$

The general solution to this type of PDE can be found using various methods such as separation of variables, Fourier series, or the method of characteristics.

Example 1:

Consider the PDE:

$$\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

This PDE represents a homogeneous second-order PDE with constant coefficients. The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$, which has a repeated root $\lambda = -2$. Therefore, the general solution is given by:

$$u(x, y) = (Ax + By)e^{-2x}$$

where A and B are arbitrary constants.

Example 2:

Consider the PDE:

$$\frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = e^{x+y}$$

Solution:

This PDE represents an inhomogeneous second-order PDE with constant coefficients. To solve it, we first find the complementary solution by solving the associated homogeneous equation. Then, we find a particular solution using a suitable method such as the method of undetermined coefficients or variation of parameters.

More Examples of Second-Order PDEs:

(a) **Poisson's Equation:**

- Poisson's equation is a second-order elliptic partial differential equation and is given by:

$$\nabla^2 u = f(x, y, z)$$

where $u(x, y, z)$ is the unknown function, and $f(x, y, z)$ is a given function.

- **Solution:** The solution to Poisson's equation depends on the boundary conditions and the domain of interest. In many cases, numerical methods such as finite difference, finite element, or spectral methods are used for solving this equation.

(b) **Hyperbolic Equations:**

- Hyperbolic equations describe wave-like phenomena and are given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- **Solution:** Solutions to hyperbolic equations typically involve specifying initial conditions or boundary conditions. Methods such as the method of characteristics or Fourier transform techniques are often employed to find solutions.

(c) **Elliptic Equations:**

- Elliptic equations arise in problems involving steady-state conditions and are given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- **Solution:** Solutions to elliptic equations depend on the boundary conditions. Techniques such as separation of variables, Fourier series, or numerical methods like finite difference or finite element methods are used for solving such equations.

Monge's Method (Method of Characteristics):

Monge's method is a technique used to solve first-order partial differential equations (PDEs) by transforming them into a system of ordinary differential equations (ODEs) along characteristic curves.

Consider a first-order PDE of the form:

$$F(x, y, u, p, q) = 0$$

where $u(x, y)$ is the unknown function, and p and q are auxiliary variables introduced to transform the PDE into a system of ODEs.

Monge's method involves solving the system of ODEs:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{F_u} = \frac{dp}{F_p} = \frac{dq}{F_q}$$

where a, b, F_u, F_p, F_q are partial derivatives of F with respect to their respective variables.

The solution $u(x, y)$ can then be expressed implicitly in terms of x, y, p, q using the relations obtained from integrating the ODEs.

Example:

Consider the first-order PDE:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Solution:

Using Monge's method, we introduce auxiliary variables p and q such that:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u} = \frac{dp}{1} = \frac{dq}{1}$$

Solving these equations, we obtain the relations:

$$u = \frac{p}{q}, \quad p = C_1 xy, \quad q = C_2$$

where C_1 and C_2 are constants of integration.

Thus, the solution to the PDE is given implicitly by:

$$u = \frac{C_1 xy}{C_2}$$