A **differential equation** is an equation involving a function and its derivatives. It can be written in the form:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

where y is the unknown function of x, and $y', y'', \ldots, y^{(n)}$ denote its derivatives with respect to x up to order n.

Exact Differential Equation:

An *exact differential equation* is a type of differential equation that can be expressed in the form:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

where $M(x, y)$ and $N(x, y)$ are functions of two variables x and y, and their first partial derivatives with respect to y and x, respectively, are equal, i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example:

Consider the differential equation:

$$(2x + 3y)\,dx + (x - 2y)\,dy = 0$$

Here, $M(x, y) = 2x + 3y$ and $N(x, y) = x - 2y$. To check for exactness, we find their partial derivatives:

$$\frac{\partial M}{\partial y} = 3$$
$$\frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. **Homogeneous Differential Equation:**

A *homogeneous differential equation* is a differential equation that can be expressed in the form:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

where $M(x, y)$ and $N(x, y)$ are functions of two variables x and y, and they are homogeneous functions of the same degree.

Example:

Consider the differential equation:

$$(x^2 + y^2)\,dx - xy\,dy = 0$$

Here, $M(x, y) = x^2 + y^2$ and $N(x, y) = -xy$. To check for homogeneity, let's substitute $x = \lambda x'$ and $y = \lambda y'$, where λ is a constant:

$$\begin{aligned} M(\lambda x', \lambda y') &= (\lambda x')^2 + (\lambda y')^2 \\ &= \lambda^2(x'^2 + y'^2) \\ &= \lambda^2 M(x', y') \end{aligned}$$

Similarly,

$$\begin{aligned} N(\lambda x', \lambda y') &= -\lambda x' \lambda y' \\ &= -\lambda^2 xy \\ &= \lambda^2 N(x', y') \end{aligned}$$

Since $M(\lambda x', \lambda y') = \lambda^2 M(x', y')$ and $N(\lambda x', \lambda y') = \lambda^2 N(x', y')$, the equation is homogeneous.

Linear First-Order Differential Equation:

A *linear first-order differential equation* is a differential equation that can be expressed in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only.

Example:

Consider the differential equation:

$$\frac{dy}{dx} + 2xy = e^x$$

This is a linear first-order differential equation with $P(x) = 2x$ and $Q(x) = e^x$. **First-Order Differential Equation with Higher Degree:**

A first-order differential equation with higher degree typically refers to equations where the highest derivative of the dependent variable is of order higher than 1. An example is:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

This is a first-order differential equation with a second-degree derivative term.

Example:

Consider the differential equation:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

This equation can be rewritten as:

$$y'' + xy' - y = 0$$

where y'' denotes the second derivative of y with respect to x, and y' denotes the first derivative of y with respect to x. **Linear Differential Equation of Order Greater Than 1:**

A *linear differential equation* of order greater than 1 is a differential equation that can be expressed in the general form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

where $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $F(x)$ are functions of x only, y is the dependent variable, and $\frac{d^ky}{dx^k}$ represents the kth derivative of y with respect to x.

Example:

Consider the linear differential equation of second order:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x$$

This equation can be expressed in the general form with $n = 2$, $a_2(x) = 1$, $a_1(x) = 2$, $a_0(x) = 1$, and $F(x) = e^x$. **Example with Solution:**

Consider the linear differential equation of second order:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Solution:

To solve this differential equation, we first find the characteristic equation:

$$r^2 - 3r + 2 = 0$$

The roots of this quadratic equation are $r_1 = 1$ and $r_2 = 2$.

Therefore, the general solution of the differential equation is given by:

$$y(x) = C_1e^x + C_2e^{2x}$$

where C_1 and C_2 are arbitrary constants determined by initial conditions or boundary conditions. **Example with Solution:**

Consider the linear differential equation of third order:

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

Solution:

To solve this differential equation, we first find the characteristic equation:

$$r^3 - 2r^2 + r - 1 = 0$$

This can be factored as $(r - 1)^3 = 0$, so the root $r = 1$ has a multiplicity of 3.

Therefore, the general solution of the differential equation is given by:

$$y(x) = (C_1 + C_2x + C_3x^2)e^x$$

where C_1, C_2, and C_3 are arbitrary constants determined by initial conditions or boundary conditions. **Example with Solution:**

Consider the linear differential equation of fourth order:

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

Solution:

To solve this differential equation, we first find the characteristic equation:

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$$

This equation can be factored as $(r-1)^4 = 0$, so the root $r = 1$ has a multiplicity of 4.

Therefore, the general solution of the differential equation is given by:

$$y(x) = (C_1 + C_2x + C_3x^2 + C_4x^3)e^x$$

where C_1, C_2, C_3, and C_4 are arbitrary constants determined by initial conditions or boundary conditions. **Types of Matrices:**

1. **Row Matrix:**

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

2. **Column Matrix:**

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

3. **Square Matrix:**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

4. **Diagonal Matrix:**

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

5. **Identity Matrix:**

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

6. **Zero Matrix:**

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Rank of a Matrix:

The *rank* of a matrix is the maximum number of linearly independent rows or columns in the matrix. It is denoted by $\text{rank}(A)$.

Example:

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

To find the rank of matrix A, we can perform row operations to reduce it to row-echelon form or reduced row-echelon form.

For matrix A, we can see that the third row is a multiple of the first row. So, the rank of matrix A is $\text{rank}(A) = 2$.

Example with Solution:

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

To find the rank of matrix A, we can perform row operations to reduce it to row-echelon form or reduced row-echelon form.

First, we subtract twice the first row from the second row and thrice the first row from the third row to get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the second and third rows are identical, we only have two linearly independent rows. Therefore, the rank of matrix A is $\text{rank}(A) = 2$.

Eigenvalues and Eigenvectors:

Eigenvalues and eigenvectors are important concepts in linear algebra. Given a square matrix A, an eigenvector v and its corresponding eigenvalue λ satisfy the equation:

$$Av = \lambda v$$

- **Eigenvalue**: An eigenvalue λ of a matrix A is a scalar such that there exists a non-zero vector v satisfying $Av = \lambda v$.
- **Eigenvector**: An eigenvector v of a matrix A is a non-zero vector that remains in the same direction after transformation by the matrix A.

Example:

Consider the matrix:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

To find the eigenvalues λ, we solve the characteristic equation $|A - \lambda I| = 0$, where I is the identity matrix.

The characteristic equation becomes:

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

Solving this quadratic equation, we find eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$.

For each eigenvalue, we find the corresponding eigenvector v by solving the equation $(A - \lambda I)v = 0$.

For $\lambda_1 = 1$, we have:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} v_1 = 0$$

Solving this system, we find the eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = 3$, we have:

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v_2 = 0$$

Solving this system, we find the eigenvector $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Thus, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$, and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

More Examples of Eigenvalues and Eigenvectors:

Example:

Consider the matrix:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

To find the eigenvalues λ, we solve the characteristic equation $|A - \lambda I| = 0$, where I is the identity matrix.

The characteristic equation becomes:

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = (\lambda - 2)(\lambda - 4) = 0$$

Solving this equation, we find eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$.

For each eigenvalue, we find the corresponding eigenvector v by solving the equation $(A - \lambda I)v = 0$.

For $\lambda_1 = 2$, we have:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v_1 = 0$$

Solving this system, we find the eigenvector $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Similarly, for $\lambda_2 = 4$, we have:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v_2 = 0$$

Solving this system, we find the eigenvector $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$, and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Cayley-Hamilton Theorem:

The Cayley-Hamilton theorem states that every square matrix A satisfies its own characteristic equation.

If A is an $n \times n$ matrix and $p(\lambda)$ is its characteristic polynomial, then the Cayley-Hamilton theorem states that $p(A) = 0$, where 0 is the zero matrix.

In other words, substituting the matrix A into its characteristic polynomial yields the zero matrix.

Example:

Consider a 2×2 matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The characteristic polynomial of A is given by:

$$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right) = (\lambda - a)(\lambda - d) - bc$$

According to the Cayley-Hamilton theorem, we have:

$$p(A) = (A - aI)(A - dI) - bcI = 0$$

Expanding and simplifying this expression, we get:

$$(A - aI)(A - dI) - bcI = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This verifies the Cayley-Hamilton theorem for a 2×2 matrix.